MZV

Introduction to multiple zeta values

§0. Intro

§1. double shuffle relation ($\leftarrow$ combinatorics)

§2. associator relations ($\leftarrow$ geometry of $M_{0,4}$, $M_{0,5}$)

§3. K-theory relation (sketch) ($\leftarrow$ motivic)

§4. braided tensor cat, qtqHQUE $\mathfrak{d}_{g}$ (slides)

(alg geom) $\xrightarrow{\text{cohom}}$ (linear alg) ; (motive)

遠アーベル	近アーベル
anabelian geom	today
profinite	pro-algebraic
Galois cat.	Tannakian cat.
$\widehat{GT}$	$\underline{GT}$
profinite π_1	Malcev π_1
far from cohom.	related to cohom.

π_1^{M}

π_1^{B}, π_1^{dR}, π_1^{et}, π_1^{crys}

related to motif

$H^* \rightsquigarrow K^*$ (ban construction)

$\pi_1 \rightsquigarrow BK^*$

§0. Intro

$k_1, \ldots, k_d \in \mathbb{Z}$

$\zeta(k_1, \ldots, k_d) = \ldots$ n

MZV

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§1. double shuffle relation (← combinatorics)

§2. associator relations (← geometry of $M_{0,4}, M_{0,5}$)

§3. K-theory relation (sketch) (← motivic)

§4. braided tensor cat, qtqHQVE alg. (slides)

(alg. geom) $\xrightarrow{\text{cohom}}$ (linear alg.)

(motivic)

遠アーベル

anabelian geom

profinite

Galois cat.

$\widehat{GT}$

profinite π_1

fun. gr. cohom.

§0. Intro

$k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} \in \mathbb{R} \cup \{\infty\}$$

abs. conv. $\iff k_d \geq 2$

Rem. Some people use the convention

of

$d=1$: Riemann ... ues

, $M_{0,S}$)

c

$$\begin{pmatrix}\text{alg.}\\ \text{geom}\end{pmatrix} \xrightarrow{\text{cohom}} \begin{pmatrix}\text{linear}\\ \text{alg.}\end{pmatrix}$$

(motivic)

遠アーベル	近アーベル	today
anabelian geom		
profinite	pro-algebraic	related to motif
Galois cat.	Tannakian cat.	
$\widehat{GT}$	$\underline{GT}$	
profinite π_1	Malcev π_1	
far from cohom	related to cohom.	

$\pi_1^{\mathcal{M}}$

π_1^{B}, π_1^{dR}, π_1^{et}, π_1^{crys}

$H^* \leftrightsquigarrow K^*$ for ζ-motiv.

$\pi_1 \sim B^{\cdot}K^*$

Rem Some people use the convention of $\sum_{n_1 > \cdots > n_d}$

d=1 : Riemann zeta values

Why MZVs ? What is interesting?

① They are related to many areas of ma…

e.g. conformal field theory, …

②

§0. Intro

$k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$

$$\zeta(k_1, \ldots, k_d) = \sum_{n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} \in \mathbb{R} \cup \{\infty\}$$

abs. conv. $\Longleftrightarrow k_d \geq 2$

d : depth

$k_1 + \cdots + k_d$: weight ($\times(-2) \sim$ wt of motives)

Rem. Some people use the convention of $\sum_{n_1 > \cdots > n_d}$

$d = 1$: Riemann zeta values

Why MZVs ? What is interest

② $\exists$ enormous $\mathbb{Q}$-linear relations among MZVs

e.g. $\zeta(3) = \zeta(1,2)$ (Euler)

$\zeta(4) = \zeta(1,1,2) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2)$

Euler except ... $= \pi^4$

$\in \mathbb{R}^{u}\{\infty\}$

$n_d^{k_d}$

Rem: Some people use the convention of $\sum_{n_1 > \cdots > n_d}$

$d=1$: Riemann zeta values

Why MZVs? What is interesting?

① They are related to many areas of math (& physics)

e.g. conformal field theory, KZ eq., moduli of curves, alg. K-theory, knot invariants, Grothendieck-Teich. gp, representation theory, graph cohom.

Z

... of (motives)

MZV space

$n>0 \quad Z_n = \left\langle \zeta(k_1, \ldots, k_d) \;\middle|\; \begin{array}{l} k_1 + \cdots + k_d = n \\ k_1, \ldots, k_{d-1} \geq 1,\ k_d \geq 2 \\ d \geq 1 \end{array} \right\rangle_{\mathbb{Q}} \subset \mathbb{R}$

$\mathbb{Q}$-span

$Z_0 = \mathbb{Q}$

e.g. $Z_0 = \mathbb{Q}$

$Z_1 = 0$

$Z_3 = \zeta(3)\mathbb{Q} + \zeta(1,2)\mathbb{Q}$

$\frac{4}{3}\zeta(2,2)$

② $\exists$ enormous $\mathbb{Q}$-linear relations among MZVs

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$\zeta(4) = \zeta(1,1,2) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2)$

Euler except $\zeta(1,1,2)$ $= \frac{\pi^4}{90}$

⋮

Z_{10} 256 index sets
dim ≤ 7

⋮

Z_{20} 262144 index sets
dim ≤ 114

MZV space

$n>0$ $Z_n = \left\langle \zeta(k_1, \ldots, k_d) \,\middle|\, \begin{matrix} k_1 + \cdots + k_d = n \\ k_1, \ldots, k_{d-1} \geq 1,\ k_d \geq 2 \\ d \geq 1 \end{matrix} \right\rangle_{\mathbb{Q}} \subset \mathbb{R}$

$\mathbb{Q}$-span

$Z_0 := \mathbb{Q}$

e.g. $Z_0 = \mathbb{Q}$ dim = 1

$Z_1 = 0$ dim = 0

$Z_2 = \zeta(2)\mathbb{Q} = \pi^2\mathbb{Q}$ dim = 1

$n_1 > \cdots > n_d$

$d=1$: Riemann zeta values

Why MZVs? What is interesting?

① T

e.g.

$(2,2)$

MZV space

$n > 0$ $\quad Z_n = \left\langle \zeta(k_1, \ldots, k_d) \,\middle|\, \begin{array}{l} k_1 + \cdots + k_d = n \\ k_1, \ldots, k_{d-1} \geq 1,\ k_d \geq 2 \\ d \geq 1 \end{array} \right\rangle_{\mathbb{Q}} \subset \mathbb{R}$

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$Z_2 = \zeta(2)\mathbb{Q} = \pi^2\mathbb{Q}$ $\quad \dim = 1$

$Z_3 = \zeta(3)\mathbb{Q} + \zeta(1,2)\mathbb{Q} = \zeta(3)\mathbb{Q}$ $\quad \dim = 1$

$Z_4 = \zeta(4)\mathbb{Q} + \zeta(1,3)\mathbb{Q} + \zeta(2,2)\mathbb{Q} + \zeta(1,1,2)\mathbb{Q} = \pi^4\mathbb{Q}$ $\quad \dim = 1$

$Z_5 = \zeta(5)\mathbb{Q} + \pi^2\zeta(3)\mathbb{Q}$ $\quad \dim \leq 2$

ζ indep. ... conjecturally $\dim = 2$

Zagier)

① They are related to many areas of math (& physics)

e.g. conformal field theory, KZ eq., moduli of curves, alg. K-theory, knot invariants, Grothendieck-Teich. $\mathcal{R}$, representation theory, graph cohom...

Z_{10} 256 index sets
$\dim \leq 7$

Z_{20} 262144 index sets
$\dim \leq 114$

the difference grows exponentially.

<u>Conj</u> (Zagier)

$d_0 = 1,\ d_1 = 0,\ d_2 = 1,$
$d_{n+3} := d_{n+1} + d_n \quad (n \geq 0)$

$\Rightarrow \dim_{\mathbb{Q}} Z_n \underset{?}{=} d_n$ for ${}^\forall n \geq 0$

(K-theoretic meaning of $\{d_n\}_n \rightsquigarrow$ §3)

<u>Th</u> (Goncharov, Terasoma, Deligne

$\dim_{\mathbb{Q}} Z_n \leq d_n$

($\sim {}^\exists$ enormous $\mathbb{Q}$-lin

<u>Rem</u> ① ${}^\exists$ "twisted version
the K-

② a part of the g
in

③ ${}^\exists$ p-adic analogue
of Conj, Th, Rem ①, ②
Furu

$n > 0$ $Z_n := \langle \ldots$ $\mid k_1, \ldots, k_{d-1} \geq 1,\ k_d \leq c,\ d \geq 1$

$Z_0 := \mathbb{Q}$

<u>e.g.</u> $Z_0 = \mathbb{Q}$ $\dim = 1$
$\dim = 0$

...gies)

$d_1 := 0,\ d_2 = 1,$

$= d_{n+1} + d_n \quad (n \geq 0)$

$Z_n \overset{?}{=} d_n$ for $\forall n \geq 0$

...etic meaning of $\{d_n\}_n$ $\rightsquigarrow$ §3)

__Th__ (Goncharov, Terasoma, Deligne-Goncharov)

$$\dim_{\mathbb{Q}} Z_n \leq d_n$$

($\rightsquigarrow$ $\exists$ enormous $\mathbb{Q}$-lin. rel'n among MZVs)

MZV $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$\mathbb{P}^1 \setminus \{0,\infty\} \cup \mu_N$

__Rem__ ① $\exists$ "twisted version" (multiple L-values) of Th (← Deligne-Goncharov)

the K-theoretic upper bound is not best possible

(N: prime, $\neq$ [illegible] $\Rightarrow \dim_{\mathbb{Q}} \lneq$ (K-theoretic est.))

② a part of the gap is related to the space of modular forms $S_2(\Gamma_1(N))$ (Goncharov)

$n > 0$ $\quad Z_n = \langle$ [illegible] $\mid k_1, \ldots, k_{d-1} \geq 1,\ k_d \geq 2,\ d \geq 1 \rangle_{\mathbb{Q}}$ — [illegible]

$\mathbb{Q}$-span

$Z_0 := \mathbb{Q}$

__e.g.__ $Z_0 = \mathbb{Q}$ $\quad \dim = 1$

$Z_1 = 0$ $\quad \dim = 0$

$\zeta(2)\mathbb{Q} = \pi^2\mathbb{Q}$ $\quad \dim = 1$

$Z_3 = \zeta(3)\mathbb{Q} + \zeta(1,2)\mathbb{Q} = \zeta(3)\mathbb{Q}$ $\quad \dim = 1$

$Z_4 = \zeta(4)\mathbb{Q} + \zeta(1,3)\mathbb{Q} + \zeta(2,2)\mathbb{Q} + \zeta(1,1,2)\mathbb{Q} = \pi^4\mathbb{Q}$ $\quad \dim = 1$

$Z_5 = \zeta(5)\mathbb{Q} + \pi^2\zeta(3)\mathbb{Q}$

ζ [illegible] $\quad \dim \leq 2$

conjecturally $\dim = 2$