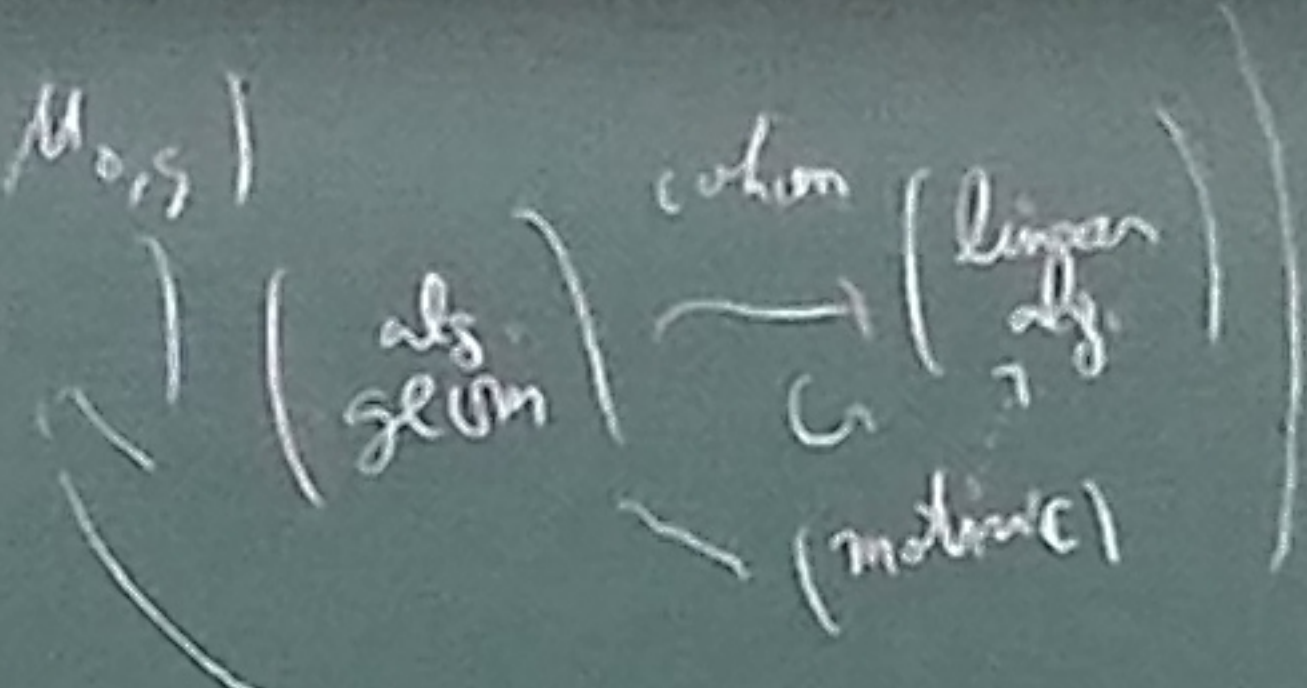




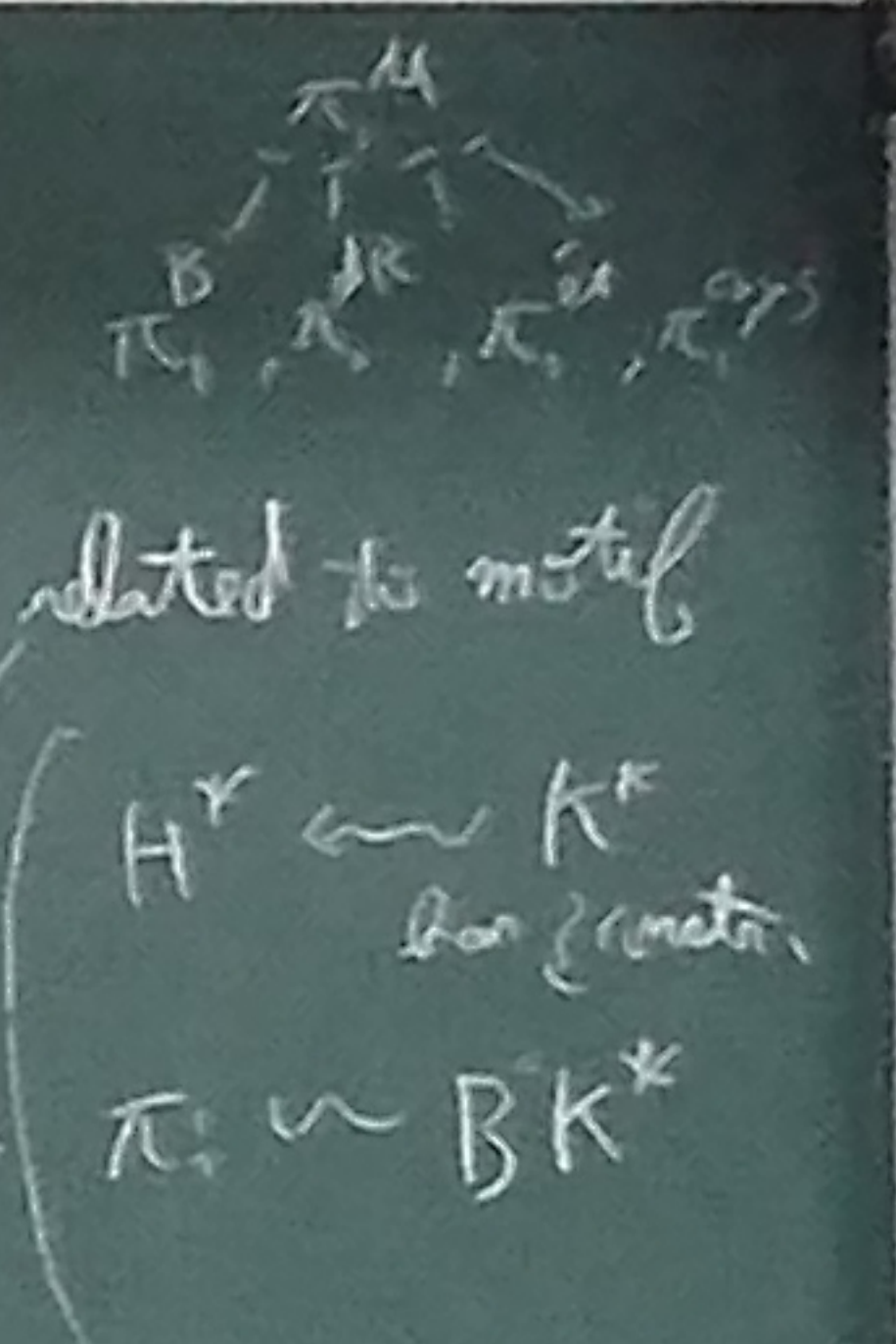
Introduction to multiple zeta values ^{MZV}

- §0. Intro
- §1. double shuffle relation (← combinatorics)
- §2. associator relations (← geometry of Mod. Mod.)
- §3. K-theory relation (sketch) (← motivic)
- §4. braided tensor cat, qta Hodge th., (slides)



\widehat{GT} P-ADIC
 commutative geom
 profinite
 Galois cat.
 \widehat{GT}
 profinite π
 far from cohom.

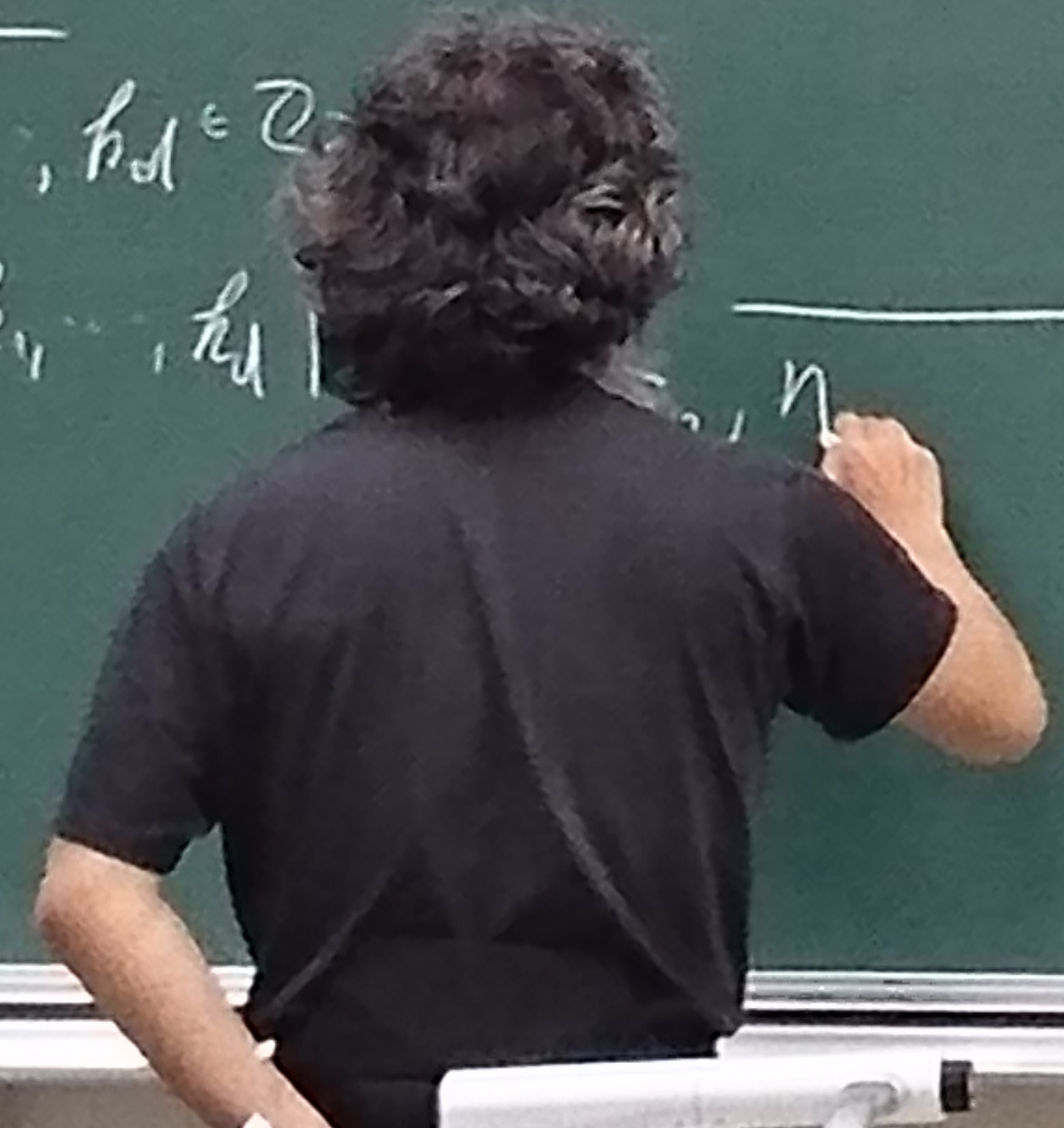
\widehat{GT} P-ADIC
 today
 pro-algebraic
 Tannakian cat.
 \widehat{GT}
 Milnor π
 related to cohom.

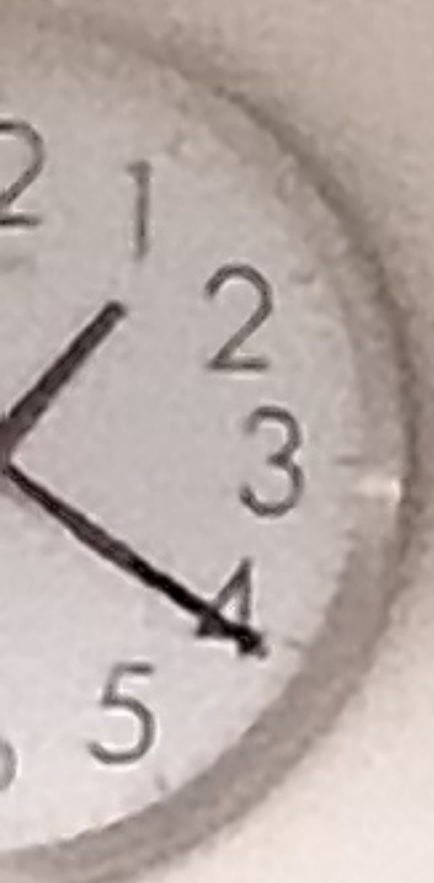


§0. Intro

$k_1, \dots, k_d \in \mathbb{Z}$

$\{k_1, \dots, k_d\}$





MZV

Introduction to multiple zeta values

§0, Intro

§1, double shuffle relation

§2, associator relations

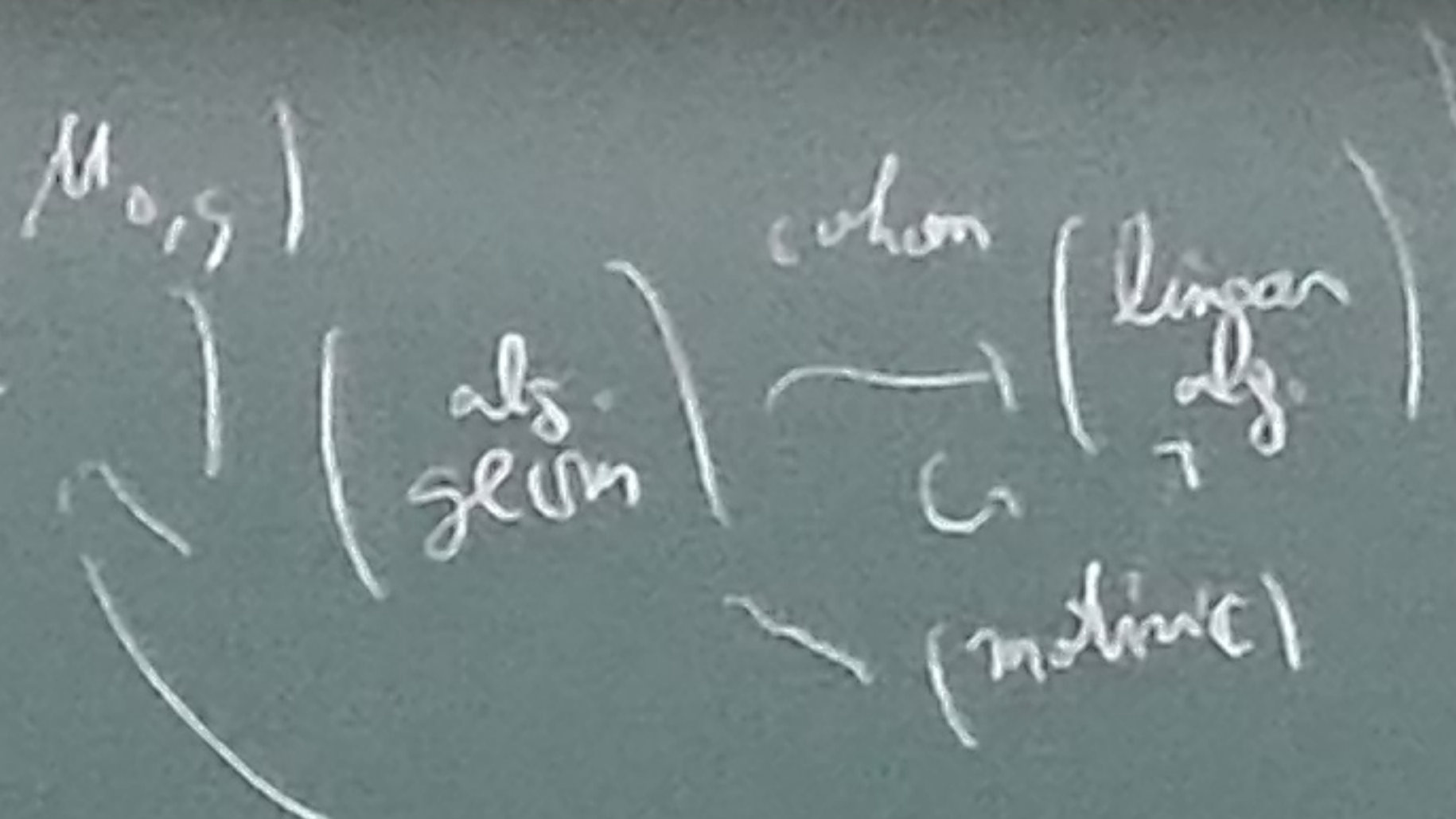
§3, K-theory relation (sketch)

§4, braided tensor cat, qtgHQVE alg, (slides)

(← combinatorics)

(← geometry of $M_{0,4}, M_{0,5}$)

(← motivic)



non-abelian
geom
profinite
Galois cat.
 \widehat{GT}
profinite π_1
far from cohom.

§0, Intro

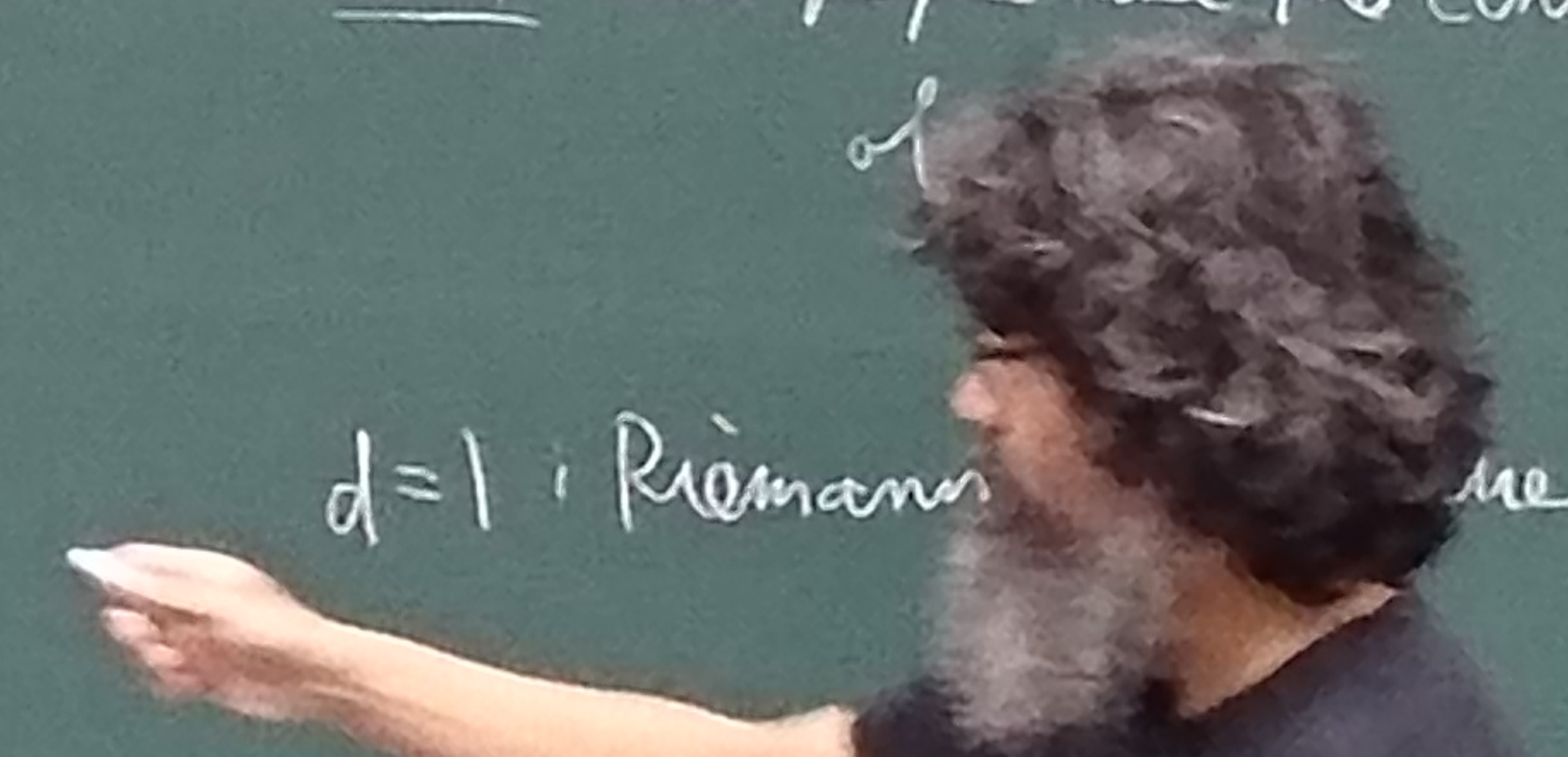
$$k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$$

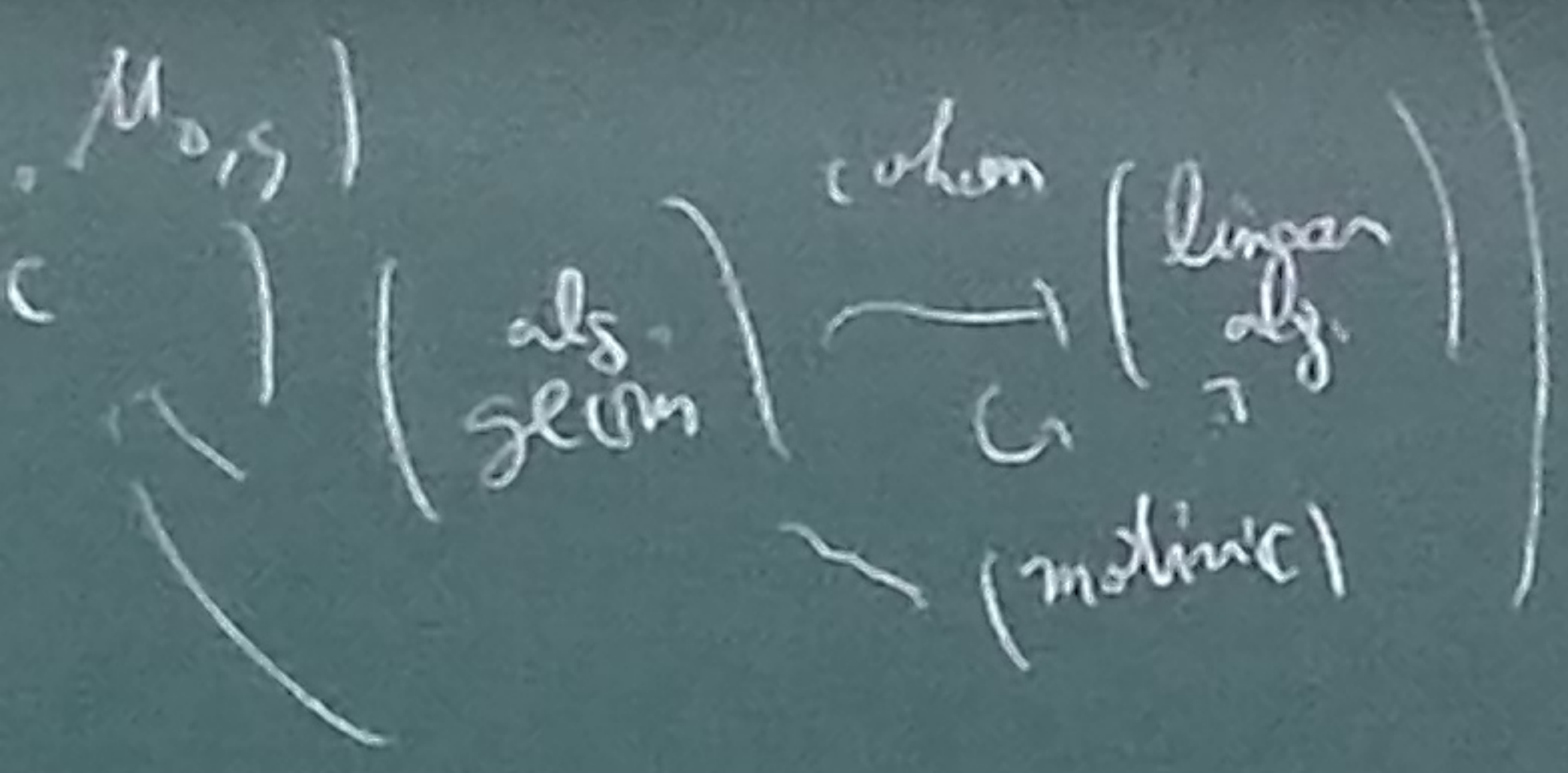
$$\zeta(k_1, \dots, k_d) = \sum_{n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \dots n_d^{k_d}} \in \mathbb{R}^u \text{ (asy)}$$

abs. conv. $\Leftrightarrow k_d \geq 2$

Remark Some people use the convention of

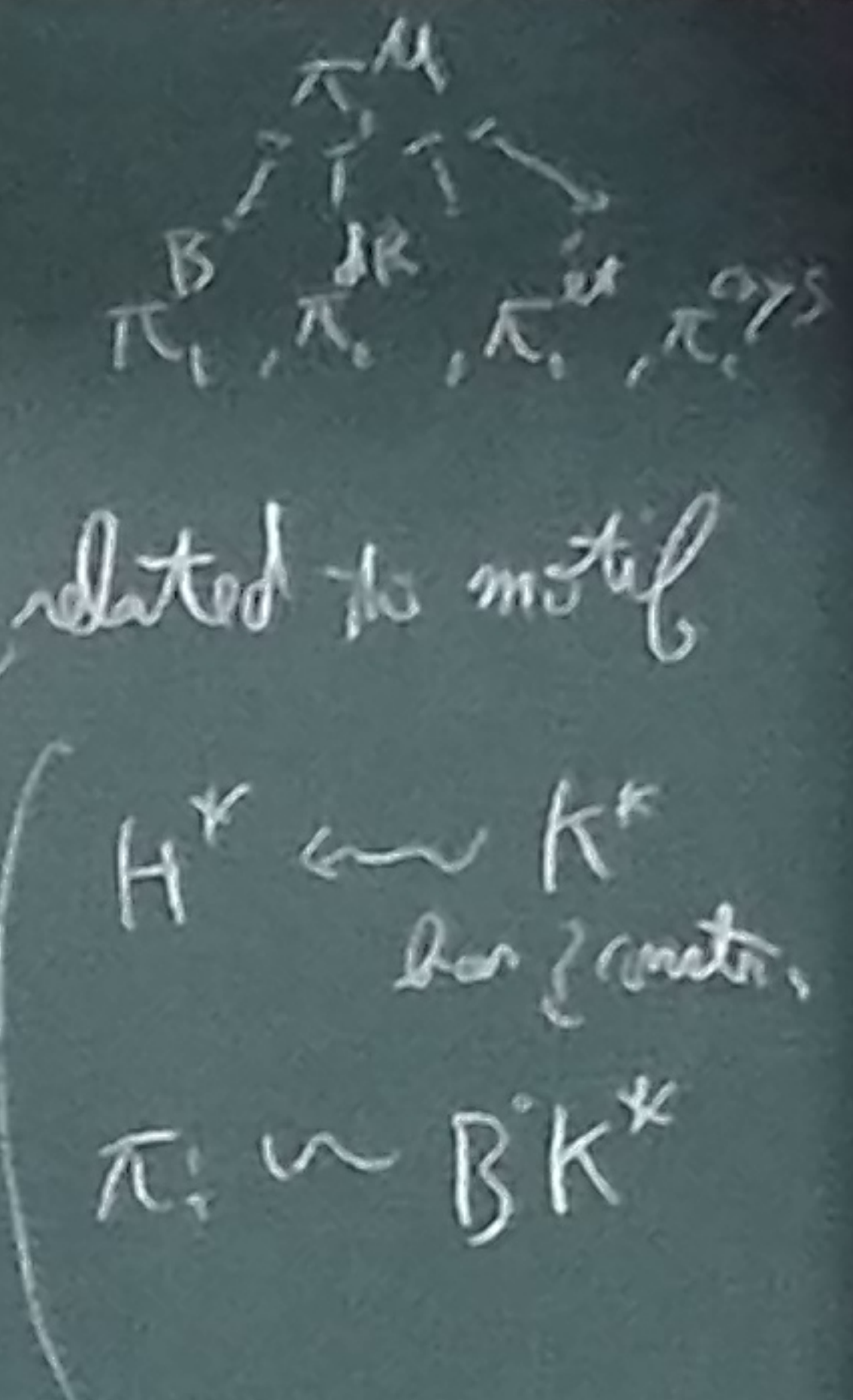
$d=1$: Riemann ζ





$\mathbb{Z}/\ell \mathbb{Z}$ P - \mathbb{A}^1
 amabelian
 geom
 profinite
 Galois cat.
 \widehat{G}
 profinite π_1
 far from cohom.

$\mathbb{Z}/\ell \mathbb{Z}$ P - \mathbb{A}^1
 today
 pro-algebraic
 Tannakian cat.
 \widehat{G}
 Mal'cev π_1
 related to cohom.



Remark: some people use the convention
 of $\sum_{n > 2d}$

$d=1$: Riemann zeta values

Why MZVs? What is interesting?

① They are related to many areas of math
 eg. conformal field theory,

②

§ 0, Intro

$$k_1, \dots, k_d \in \mathbb{Z}_{\geq 1}$$

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \dots n_d^{k_d}} \in \mathbb{R}^u \text{ (asy)}$$

also, conv. $\Leftrightarrow k_d \geq 2$

d : depth

$k_1 + \dots + k_d$: weight ($(1,2) \sim$ not of motives)

Rem. some people use the convention

$$\sum_{n_1 < \dots < n_d}$$

$d=1$: Riemann zeta values

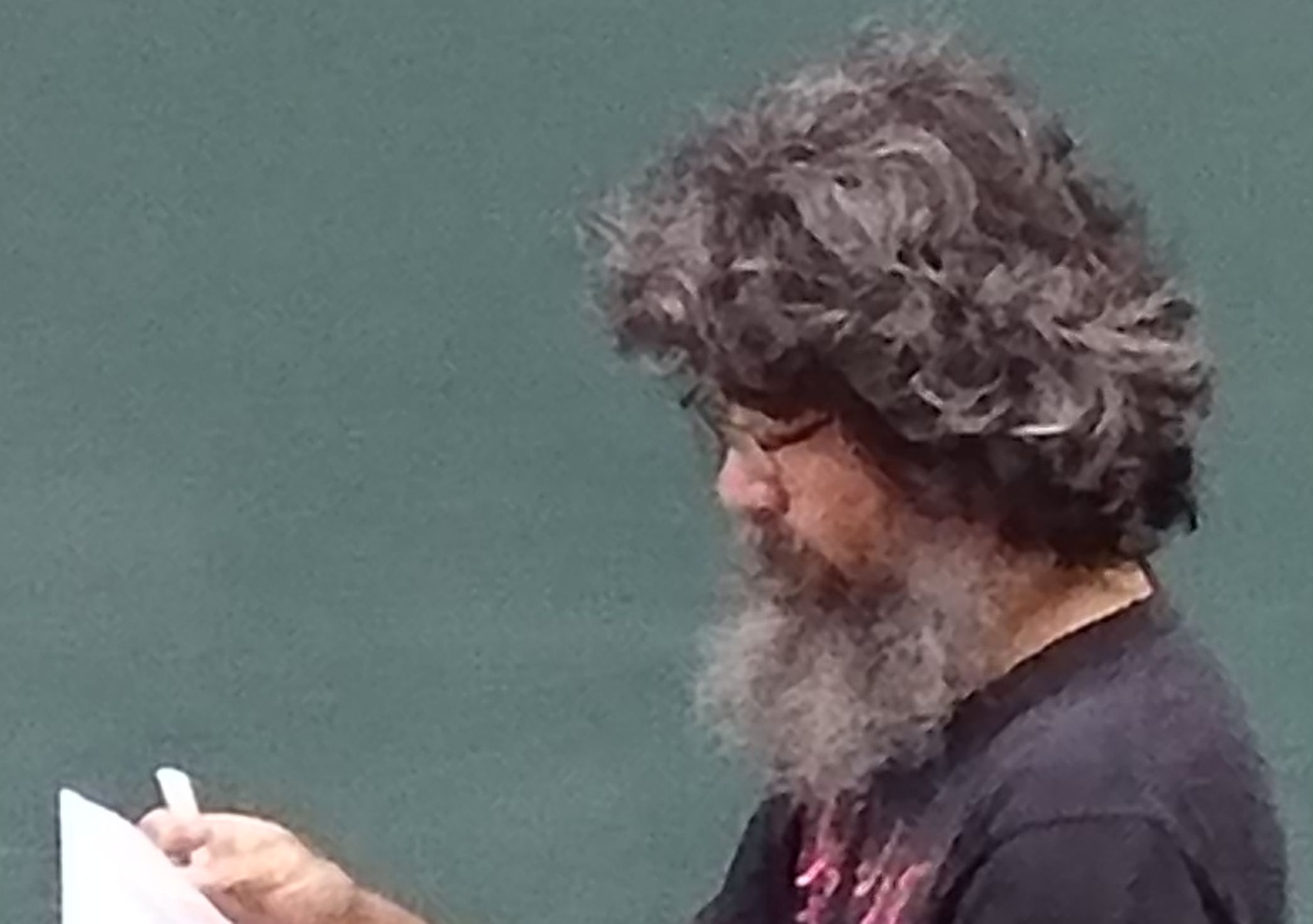
Why MZVs? What is interesting

② \Rightarrow enormous \mathbb{Q} -linear relations among MZVs

0.9. $\zeta(3) = \zeta(1,2)$ (Euler)

$$\zeta(4) = \zeta(1,1,2) = 4 \zeta(1,3) = \frac{4}{3} \zeta(2,2)$$

Euler constant $\omega_1 = \pi^4$



$$\frac{b_d}{n_d} \in \mathbb{R}^u \text{ for } y$$

2
not of nature

From some people use the convention

$$d \sum_{n, > 2nd}$$

$d=1$: Riemann zeta values

Why MZVs? What is interesting?

① They are related to many areas of math. (& physics)

e.g. conformal field theory, KZ eq.,
moduli of curves, Dyk theory,
finite invariants, Grothendieck-Teich.
representation theory, graph cohom.

MZV space

$$n > 0 \quad \mathbb{Z}_n = \left\langle \left\{ (k_1, \dots, k_d) \mid \begin{array}{l} k_1 + \dots + k_d = n \\ k_i \geq 1, k_d \geq 2 \\ d \geq 1 \end{array} \right\} \right\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

\mathbb{Q} -span

$$\mathbb{Z}_0 = \mathbb{Q}$$

e.g. $\mathbb{Z}_0 = \mathbb{Q}$

$$\mathbb{Z}_1 = \mathbb{Q}$$

$$\mathbb{Z}_3 = \mathbb{Q}(3) \oplus \mathbb{Q}(1) \oplus \mathbb{Q}$$

$$\frac{4}{3} \mathbb{Q}(2, 2)$$



② \Rightarrow enormous \mathbb{Q} -linear relations among MZVs

e.g. $\zeta(3) = \zeta(1,2)$ (Euler)
 $\zeta(4) = \zeta(1,1,2) = 4 \zeta(1,3) = \frac{4}{3} \zeta(2,2)$
 Euler except $\zeta(1,1,2) = \frac{\pi^4}{90}$

MZV space

$$n > 0 \quad Z_n = \left\langle \zeta(k_1, \dots, k_d) \mid \begin{array}{l} k_1 + \dots + k_d = n \\ k_i \geq 1, k_d \geq 2 \\ d \geq 1 \end{array} \right\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

\mathbb{Q} -span

$Z_0 = \mathbb{Q}$

e.g. $Z_0 = \mathbb{Q} \quad \dim = 1$
 $Z_1 = \mathbb{Q} \quad \dim = 0$
 $Z_2 = \zeta(2)\mathbb{Q} = \pi^2\mathbb{Q} \quad \dim = 1$

Z_{10} 256 inden sets
 $\dim \leq 7$

Z_{20} 262144 inden sets
 $\dim \leq 114$

$n \leftarrow n$
 $\text{For } n < n$

$d=1$: Riemann zeta values

When MZVs? What is interesting?

① T

e.g.

MZV space

$$n > 0 \quad Z_n = \left\langle \zeta(k_1, \dots, k_d) \mid \begin{array}{l} k_1 + \dots + k_d = n \\ k_i \geq 1, k_d \geq 2 \\ d \geq 1 \end{array} \right\rangle_{\mathbb{Q}} \subset \mathbb{R}$$

\mathbb{Q} -span

$$Z_0 = \mathbb{Q}$$

e.g.

$$\begin{aligned} Z_0 &= \mathbb{Q} & \dim &= 1 \\ Z_1 &= 0 & \dim &= 0 \\ Z_2 &= \zeta(2)\mathbb{Q} = \pi^2\mathbb{Q} & \dim &= 1 \end{aligned}$$

$$Z_3 = \zeta(3)\mathbb{Q} + \zeta(1,2)\mathbb{Q} = \zeta(3)\mathbb{Q} \quad \dim = 1$$

$$Z_4 = \zeta(4)\mathbb{Q} + \zeta(1,3)\mathbb{Q} + \zeta(2,2)\mathbb{Q} + \zeta(1,1,2)\mathbb{Q} = \pi^4\mathbb{Q} \quad \dim = 1$$

$$Z_5 = \zeta(5)\mathbb{Q} + \pi^2\zeta(3)\mathbb{Q}$$

ζ index str conjecturally $\dim = 2$

zajier)

① They are related to many areas of math. (& physics)

e.g. conformal field theory, KZ eq., moduli of curves, alg. K-theory, finite invariants, Grothendieck-Teich. \mathcal{P} representation theory, graph colouring



Z_{10} 256 indecomposables
 $d_n \leq 7$

Z_{20} 262144 indecomposables
 $d_n \leq 114$

the difference grows exponentially.

Conj (Zagier)
 $d_0 = 1, d_1 = 0, d_2 = 1,$
 $d_{n+3} = d_{n+1} + d_n \quad (n \geq 0)$
 $\Rightarrow \dim_{\mathbb{Q}} Z_n = d_n \quad \forall n \geq 0$
 (K-theoretic meaning of $\{d_n\}$)

Th (Groncharov, Terasama, Deligne)
 $\dim_{\mathbb{Q}} Z_n \leq d_n$
 \sim enormous \mathbb{Q} -lin.
Prop \exists "twisted version" of the K-

② a part of the g...
 in n

③ \exists p-adic analogue
 of Conj, Th, Prop ①, ②
 Furusawa

$n > 0 \quad Z_n = \bigcup_{d \geq 1} \dots$
 $Z_0 = \mathbb{Q}$
 e.g. $Z_0 = \mathbb{Q}$
 $Z_1 = 0$
 $\dim = 1$
 $\dim = 0$

gives
 $d_1 = 0, d_2 = 1,$
 $= d_{n+1} + d_n \quad (n \geq 0)$

$Z_n = d_n$ for $n \geq 0$

arithmetic meaning of $\{d_n\}$
 \leadsto §3

Th (Goncharov, Terasoma, Deligne-Goncharov)

$$\dim_{\mathbb{Q}} Z_n \leq d_n$$

enormous \mathbb{Q} -lin. rel'n among MZVs $\leftarrow \mathbb{P}^1(\mathbb{Q}, \omega^u, \mu_N)$

Rem \exists "twisted version" (multiple values) of Th (Deligne-Goncharov)
 \leftarrow the K -theoretic upper bound is not best possible

$$(N\text{-prim}, \chi \Rightarrow \dim_{\mathbb{Q}} \leq \left\lfloor \frac{K\text{-theoretic}}{29} \right\rfloor)$$

$\textcircled{2}$ a part of the gap is related to the space of modular forms $S_2(\Gamma_1(N))$
 (Goncharov)

$n > 0$ $Z_n = \langle \dots \rangle$

$$Z_0 = \mathbb{Q}$$

e.g.

$$Z_0 = \mathbb{Q} \quad \dim = 1$$

$$Z_1 = 0 \quad \dim = 0$$

$$Z_2 = \langle \pi^2 \rangle = \pi^2 \mathbb{Q} \quad \dim = 1$$

$\left| \begin{matrix} \tau_{d_1}, \tau_{d_2}, \dots, \tau_{d_n} \\ d \geq 1 \end{matrix} \right|_{\mathbb{Q}} = \dots$
 \mathbb{Q} -span

$$Z_3 = \langle \pi^3 \rangle_{\mathbb{Q}} + \langle \pi^2 \rangle_{\mathbb{Q}} = \langle \pi^3 \rangle_{\mathbb{Q}} \quad \dim = 1$$

$$Z_4 = \langle \pi^4 \rangle_{\mathbb{Q}} + \langle \pi^3 \rangle_{\mathbb{Q}} + \langle \pi^2 \rangle_{\mathbb{Q}} + \langle \pi \rangle_{\mathbb{Q}} = \pi^4 \mathbb{Q} \quad \dim = 1$$

$$Z_5 = \langle \pi^5 \rangle_{\mathbb{Q}} + \pi^2 \langle \pi^3 \rangle_{\mathbb{Q}}$$

$\&$ under str

conjecturally $\dim = 2$